Proofs for paper “Adding Synchronization and Rolling Shutter in Multi-Camera Bundle Adjustment”

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1 Proof for Eq. 3 in the Paper

Lemma 1.1. *Let vectors* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{k+3}$, *strictly positive reals a, b, and function*

$$D(\mathbf{x},\mathbf{y},\mathbf{z},a,b) = \frac{b\mathbf{z}}{a(a+b)} - \frac{a\mathbf{x}}{b(a+b)} + \frac{(a-b)\mathbf{y}}{ab}. \tag{1}$$

If $M : \mathbb{R} \to \mathbb{R}^{k+3}$ *is a* $\mathcal{C}^3$ *continuous function and* $t \in \mathbb{R}$,

$$M'(t) = D(M(t-b), M(t), M(t+a), a, b) + \mathcal{O}(a^2+b^2) \tag{2}$$

Proof. Since M is $\mathcal{C}^3$ continuous,

$$M(t+a) = M(t) + aM'(t) + \frac{a^2}{2}M''(t) + \mathcal{O}(a^3) \tag{3}$$

$$M(t-b) = M(t) - bM'(t) + \frac{b^2}{2}M''(t) + \mathcal{O}(b^3). \tag{4}$$

We eliminate $M''(t)$ by summing $\frac{b}{a}$(Eq. 3) -$\frac{a}{b}$(Eq. 4):

$$\frac{b}{a}M(t+a) - \frac{a}{b}M(t-b) = (\frac{b}{a} - \frac{a}{b})M(t) + (b+a)M'(t) + b\mathcal{O}(a^2) + a\mathcal{O}(b^2). \tag{5}$$

Since $a > 0$ and $b > 0$,

$$M'(t) = \frac{1}{a+b}(\frac{b}{a}M(t+a) - \frac{a}{b}M(t-b) + (\frac{a}{b} - \frac{b}{a})M(t)) + \mathcal{O}(a^2+b^2). \tag{6}$$

□

We use this lemma with $t = t_i$, $a = t_{i+1} - t_i$, $b = t_i - t_{i-1}$, $\Delta = \max_i(t_{i+1} - t_i)$, and obtain

$$M'(t_i) = D(M(t_{i-1}), M(t_i), M(t_{i+1}), t_{i+1} - t_i, t_i - t_{i-1}) + \mathcal{O}(\Delta^2). \tag{7}$$

2 The Singularities of Euler Angles (Sec. 4.1 in the Paper)

The proof of Lemma 2.1 is more detailed than that in [□]. Let $\mathcal{R}(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_x(\alpha)$ where $R_x(\alpha)$, $R_y(\beta)$ and $R_z(\gamma)$ are the rotations around the vectors of the canonical basis of $\mathbb{R}^3$ with respective angles α, β and γ. Let $\partial\mathcal{R}$ be the jacobian of $\mathcal{R}$ at (α,β,γ) with respect to (α,β,γ). We have $\partial\mathcal{R} = \begin{pmatrix} \frac{\partial R}{\partial \alpha} & \frac{\partial R}{\partial \beta} & \frac{\partial R}{\partial \gamma} \end{pmatrix} \in \mathbb{R}^{9\times 3}$.

Lemma 2.1. $\ker \partial\mathcal{R} \neq 0$ *if and only if there is* $k \in \mathbb{Z}$ *such that* $\beta = \pi/2 + k\pi$, i.e. *if and only if the coefficient on the bottom-left corner of* $\mathcal{R}(\alpha,\beta,\gamma)$ *is* 1 *or* -1.

Proof. We use shortened notations $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$, $c_\beta = \cos\beta$, $s_\beta = \sin\beta$, $c_\gamma = \cos\gamma$ and $s_\gamma = \sin\gamma$. Thus

$$\mathcal{R}(\alpha,\beta,\gamma) = \begin{pmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\beta & s_\beta s_\alpha & s_\beta c_\alpha \\ 0 & c_\alpha & -s_\alpha \\ -s_\beta & c_\beta s_\alpha & c_\beta c_\alpha \end{pmatrix} \tag{9}$$

$$= \begin{pmatrix} c_\gamma c_\beta & c_\gamma s_\beta s_\alpha - s_\gamma c_\alpha & c_\gamma s_\beta c_\alpha + s_\gamma s_\alpha \\ s_\gamma c_\beta & s_\gamma s_\beta s_\alpha + c_\gamma c_\alpha & s_\gamma s_\beta c_\alpha - c_\gamma s_\alpha \\ -s_\beta & c_\beta s_\alpha & c_\beta c_\alpha \end{pmatrix}. \tag{10}$$

First we show that $\beta = \pi/2 + k\pi$ implies $\ker \partial\mathcal{R} \neq 0$. Let $\varepsilon = 1$ if k is even, otherwise $\varepsilon = -1$. We have $s_\beta = \sin(\varepsilon\pi/2) = \varepsilon$, $c_\beta = \cos(\varepsilon\pi/2) = 0$, $\sin(\varepsilon\gamma) = \varepsilon s_\gamma$ and $\cos(\varepsilon\gamma) = c_\gamma$. Thus

$$R(\alpha,\varepsilon\pi/2,\gamma) = \begin{pmatrix} 0 & \varepsilon c_\gamma s_\alpha - s_\gamma c_\alpha & \varepsilon c_\gamma c_\alpha + s_\gamma s_\alpha \\ 0 & \varepsilon s_\gamma s_\alpha + c_\gamma c_\alpha & \varepsilon s_\gamma c_\alpha - c_\gamma s_\alpha \\ -\varepsilon & 0 & 0 \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 0 & \varepsilon\sin(\alpha - \varepsilon\gamma) & \varepsilon\cos(\alpha-\varepsilon\gamma) \\ 0 & \cos(\alpha-\varepsilon\gamma) & -\sin(\alpha-\varepsilon\gamma) \\ -\varepsilon & 0 & 0 \end{pmatrix}. \tag{12}$$

Thus $\delta \mapsto R(\alpha+\varepsilon\delta, \varepsilon\pi/2, \gamma+\delta)$ is a constant function. We derivate it at $\delta = 0$ thanks to the chain rule and obtain $\begin{pmatrix} \frac{\partial R}{\partial \alpha} & \frac{\partial R}{\partial \beta} & \frac{\partial R}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 & 1 \end{pmatrix}^T = 0$ at point (α,β,γ).

Second we show that $\beta \neq \pi/2 + k\pi$ and $\begin{pmatrix} \frac{\partial R}{\partial \alpha} & \frac{\partial R}{\partial \beta} & \frac{\partial R}{\partial \gamma} \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix}^T = 0$ imply $a = b = c = 0$. Using derivative of the first column of Eq. 10, we obtain

$$b \begin{pmatrix} -c_\gamma s_\beta \\ -s_\gamma s_\beta \\ -c_\beta \end{pmatrix} + c \begin{pmatrix} -s_\gamma c_\beta \\ c_\gamma c_\beta \\ 0 \end{pmatrix} = 0 \text{ and } c_\beta \neq 0. \tag{13}$$

Thus $b = 0$ and $c = 0$. Now we have $a\frac{\partial R}{\partial \alpha} = 0$. Using derivative of the last row of Eq. 10 and $c_\beta \neq 0$, we obtain $a = 0$. □

Lemma 2.2. *If* $\mathtt{A}$ *and* $\mathtt{B}$ *are two invertible* 3×3 *matrices,* $\ker \partial\mathcal{R} = \ker \partial(\mathtt{A}\mathcal{R}\mathtt{B})$.

Proof. Let $\mathbf{x} \in \ker \partial \mathcal{R}$, $\mathcal{R}_{ij}$ be the coefficients of $\mathcal{R}$, and $\partial \mathcal{R}_{ij}$ is the gradient of $\mathcal{R}_{ij}$ with respect to parameters (α, β, γ). Thus $\partial \mathcal{R}_{ij}.\mathbf{x} = 0$ and

$$(\partial(\mathtt{A}\mathcal{R}\mathtt{B})_{ij}).\mathbf{x} = (\partial(\sum_{k,l} \mathtt{A}_{ik}\mathcal{R}_{kl}\mathtt{B}_{lj})).\mathbf{x} = \sum_{k,l} \mathtt{A}_{ik}\mathtt{B}_{lj}(\partial \mathcal{R}_{kl}).\mathbf{x} = 0. \tag{14}$$

We see that $\partial(\mathtt{A}\mathcal{R}\mathtt{B}).\mathbf{x} = 0$, i.e. $\ker \partial \mathcal{R} \subseteq \ker \partial(\mathtt{A}\mathcal{R}\mathtt{B})$. Since $\mathtt{A}$ and $\mathtt{B}$ are invertible, we use this inclusion (replace $\mathcal{R}$ by $\mathtt{A}\mathcal{R}\mathtt{B}$, replace $\mathtt{A}$ by $\mathtt{A}^{-1}$, replace $\mathtt{B}$ by $\mathtt{B}^{-1}$) and obtain

$$\ker \partial(\mathtt{A}\mathcal{R}\mathtt{B}) \subseteq \ker \partial(\mathtt{A}^{-1}(\mathtt{A}\mathcal{R}\mathtt{B})\mathtt{B}^{-1}) = \ker \partial \mathcal{R}. \tag{15}$$

□

References

[1] P. Singla, D.Mortari, and J.L.Junkins. How to avoid singularity when using Euler angles ? In *AAS Space Flight Mechanics Conference*, 2004.